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Loop-free Compositions of Certain Finite Automata

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The binary, periodic output sequence of a certain type of composite automaton is studied. Properties of the output sequence such as frequency of symbols and periods of subpatterns are expressed by properties of the component automata and their connections.

1. INTRODUCTION

In this paper we study loop-free compositions of certain finite periodic automata of a uniform type. The mass fabrication techniques of large scale integration makes this of interest. The components will be automata acting on a triggering signal denoted by 1. If at an instant of time the triggering signal is not applied, no action occurs, i.e. there is no state transition and no output. No triggering signal at an instant of time will be denoted by the input symbol 0. We shall, thus, consider automata accepting the input alphabet $\{0, 1\}$. If also the output alphabet is $\{0, 1\}$, the output may be used as input for similar automata. No response to the input 0 is naturally interpreted as output symbol 0.

When a composite automaton made up from component automata of the type above is triggered by a clock, i.e. input sequence $(1, 1,...)$, the response sequence is binary and periodic. We shall study properties of the response sequence such as frequency of 1's and 0's and periods of subsequences expressed by simple properties of the component automata and the connections.

The paper is organized as follows: In Section 2 the component automata are described mathematically together with the response sequence to a given input sequence. In Section 3 the connections of the composite automaton are described by a connection graph. Some algebraic properties of periodic binary sequences are investigated in Section 4. These are used in Section 5 to analyze the structure of the response sequences of some classes of composite automata. An example is worked out in Section 6.

2. The Component Automata

Let A be a finite automaton with states S, input/output alphabet $\{0, 1\}$, next state function d: $\{0, 1\} \times S \to S$ and output function z: $\{0, 1\} \times S \to \{0, 1\}$ such that $d(0, s) = s$ and $z(0, s) = 0$ for all s in S. Further suppose that A is strongly connected in the sense that for all pairs of states s_i, s_j there is an input sequence which takes A from s_i to s_j. Let the number of states be n, and suppose A is in state s_0. Put $d(1, s_0) = s_1$, $z(1, s_0) = a_0$, etc. Since A is strongly connected, $d(1, s_{n-1}) = s_0$ and the autonomous response sequence is $\bar{a} = (a_0, a_1, ..., a_{n-1}, a_0, ...)$. Any equivalent automaton would generate the same response sequence $\bar{a}$ to the input sequence $\bar{1} = (1, 1, ...)$. Conversely, let p be any period for a binary sequence $\bar{a}$. Then (modulo an isomorphism of automata) there is only one automaton of the type above with p states and response sequence $\bar{a}$.

Since another choice of period for $\bar{a}$ would give an equivalent automaton, every periodic binary sequence specifies an equivalence class of automata of the type defined. We shall, therefore, not distinguish between such classes of automata with distinguished initial states and the binary periodic sequences representing them.

If $\bar{b} = (b_j)$ is any binary sequence used as input sequence to the automaton A, the response sequence to this input sequence is easily calculated as

$$c_j = b_j a_{G(j)}, \quad \text{where } G(0) = 0, \qquad G(j) = \Big(\sum_{k<j} b_k\Big) \bmod n \quad \text{if } j > 0.$$

If the sequence $\bar{b}$ is periodic, so is $\bar{c}$, and, thus, a composition between periodic binary sequences to be denoted $\bar{c} = \bar{b}\bar{a}$ is defined.

Examples. The simplest nontrivial example of an automaton of the type above is a flip–flop generating one of the sequences $(0, 1, 0, 1, ...)$ or $(1, 0, 1, 0, ...)$ depending on the initial state. If the sequence $\bar{b} = (0, 1, 1, 1, 1, 1, 1, 1, ...)$ of period 8 is used as input sequence to the automaton $\bar{a} = (0, 1, 0, 1, ...)$, the response sequence is $\bar{b}\bar{a} = (0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, ...)$ of period 16.

3. The Composite Automaton

A composite automaton A will be described by a directed graph G and a matrix $\bar{G}$. The connections between the component automata are given by the directed graph, the interconnection logics are situated at the vertices, and the component automata are given by the matrix. The component automata can be thought of as situated on the edges of the graph.

Let G be a finite, directed graph with vertices $\{v_0, v_1, \ldots, v_t\}$ and $\bar{G} = ((\bar{x}_{ij}))$ a matrix with entries periodic binary sequences $\bar{x}_{ij}$, $i, j = 0, 1, \ldots, t$ such that $\bar{x}_{ij} = (0, 0, \ldots)$ if and only if (v_i, v_j) is not an edge in G. We call $((\bar{x}_{ij}))$ a connection matrix associated with the graph G. The interconnection logics at the vertices will be addition mod 2 in the following sense: At any time instant the output symbols from the automata $\{\bar{x}_{ij}\}_i$ at the ingoing edges to vertex v_j are added mod 2, the sum being distributed as input signals to all automata $\{\bar{x}_{jk}\}_k$ at the outgoing edges from v_j. Thus, we are concerned exclusively with linear series-parallel networks. An example is given in Section 6.

Not all graphs are accepted in describing the connections of a composite automaton, and we make the following restrictions: All graphs specifying the connections of a composite automaton are supposed to be finite, directed graphs with the upper and lower lattice property. If G is such a graph with vertices $\{v_0, v_1, \ldots, v_t\}$, we suppose that the vertices are labeled such that (v_i, v_j) is never an edge if $i \geqslant j$. Such graphs are called connection graphs in this paper.

In a graph, v_0 shall always denote the initial vertex and v_t the terminal vertex. Notice that if G is any connection graph with vertices $\{v_0, \ldots, v_t\}$, the subgraph G_i consisting of all vertices and edges on all paths from v_0 to v_i in G, is a connection graph for all vertices v_i. A vertex v in a graph G is said to be on level L if there exists a path from v_0 to v in G of length L and no path of length greater than L. If the terminal vertex v_t is on level K, K is called the order of the graph. A vertex v is said to be on colevel L if there exists a path from v to v_t of length L and no path of length less than L. Let $\bar{G} = ((\bar{x}_{ij}))$ be a connection matrix associated with the graph G. With every vertex v and edge (u, w) in G we shall associate periodic binary sequences $\bar{x}(v)$, $\bar{x}(u, w)$. Put $\bar{x}(v_0) = \bar{1} = (1, 1, \ldots)$. Suppose $\bar{x}(v_j)$, $\bar{x}(v_j, v_k)$ is defined for all vertices v_j and edges (v_j, v_k) with v_j on a level less than L. Then if v_i is on level L, define $\bar{x}(v_i) = \sum_j \bar{x}(v_j, v_i)$, and $\bar{x}(v_i, v_k) = \bar{x}(v_i)\, \bar{x}_{ik}$ for all vertices v_k. It is then easy to see that the sequence $\bar{x}(v_t)$ is the periodic binary response sequence of the composite automaton when clocked regularly, i.e. with input sequence $\bar{1}$. An example is given in Section 6.

4. The Group of Periodic Binary Sequences

In this section some algebraic properties of binary periodic sequences needed in Section 5 to analyze the structure of the response sequences of certain classes of composite automata are derived. The set of periodic binary sequences is an Abelian group under addition

$$(x_0, x_1, \ldots) + (y_0, y_1, \ldots) = (x_0 + y_0, x_1 + y_1, \ldots),$$

where addition is mod 2. We shall use the notations $\bar{0} = (0, 0, \ldots)$, $\bar{1} = (1, 1, \ldots)$. Let $\bar{x}$ be a sequence and suppose p is a period for $\bar{x}$. Put $w(\bar{x}, p) = \sum_{i=0}^{p-1} x_i$, $\bar{x} = (x_0, x_1, \ldots, x_{p-1}, x_0, \ldots)$ and $\bar{w}(\bar{x}) = w(\bar{x}, p)/p$. The last definition is clearly independent of choice of period for $\bar{x}$, giving the frequency of 1's in $\bar{x}$.

Let $\bar{x} = (x_j)$ and $\bar{y} = (y_j)$ be two sequences of periods m, n, respectively. With the composite sequence $\bar{z} = \bar{x}\bar{y}$ as defined in Section 2: $z_j = x_j y_{G(j)}$, where $G(0) = 0$, $G(j) = \sum_{k<j} x_k$ if $j > 0$. $\bar{z}$ is a well defined sequence with minimal period a divisor of mn. The definition is clearly independent of choice of period for $\bar{x}$ and $\bar{y}$.

The composition $\bar{z} = \bar{x}\bar{y}$, thus defined, is associative with the following properties:

(1) $\bar{1}\bar{x} = \bar{x}\bar{1} = \bar{x}$.

(2) $\bar{x}\bar{y} = \bar{0}$ if and only if $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$.

(3) $\bar{x}(\bar{y} + \bar{z}) = (\bar{x}\bar{y}) + (\bar{x}\bar{z})$.

(4) $\bar{w}(\bar{x}\bar{y}) = \bar{w}(\bar{x})\,\bar{w}(\bar{y})$.

Suppose $\bar{x} = \bar{x}_0 + \cdots + \bar{x}_n$ and $\bar{w}(\bar{x}) = \bar{w}(\bar{x}_0) + \cdots + \bar{w}(\bar{x}_n)$. In that case $\{\bar{x}_0, \ldots, \bar{x}_n\}$ is called a splitting (of $\bar{x}$) and $\bar{x}_i$ is called a split factor in $\bar{x}$ for every i. $\{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n\}$ is a splitting if and only if, for each p and for most one i, $0 \leqslant i \leqslant n$, $(\bar{x}_i)_p = 1$ where $(\bar{x}_i)_p$ denotes the term labelled p in the sequence $\bar{x}_i$. Thus, where $\{\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n\}$ is a splitting of $\bar{x}$, if $(\bar{x})_p = 1$ then, for exactly one 1, $(\bar{x}_i)_p = 1$; and if $(\bar{x})_p = 0$ then, for all i, $(\bar{x}_i)_p = 0$. Then:

(5) $\bar{x}\bar{y}$ is a split factor in $\bar{x}$.

(6) If $\{\bar{x}_0, \ldots, \bar{x}_n\}$ is a splitting, then $\{\bar{x}_0\bar{y}_0, \ldots, \bar{x}_n\bar{y}_n\}$ is a splitting.

(7) If $\{\bar{y}_0, \ldots, \bar{y}_n\}$ is a splitting, then $\{\bar{x}\bar{y}_0, \ldots, \bar{x}\bar{y}_n\}$ is a splitting.

Let q be a fixed integer and $\bar{x}$ a sequence such that for some integer K q^K is a period of $\bar{x}$ and $w(\bar{x}, q^K)$ is prime to q. The set of all such sequences is denoted by $\bar{X}(q)$. Then:

(8) If $\bar{x}_0, \ldots, \bar{x}_n$ are in $\bar{X}(q)$, then $\bar{x}_0\bar{x}_1 \cdots \bar{x}_n$ is in $\bar{X}(q)$.

(9) Suppose $\bar{x}_0, \ldots, \bar{x}_n$ are in $\bar{X}(q)$ and $\{\bar{x}_0, \ldots, \bar{x}_n\}$ is a splitting. If there is only one sequence $\bar{x}_i$ of greatest minimal period, then

$$\bar{x}_0 + \cdots + \bar{x}_n \text{ is in } \bar{X}(q).$$

(10) Let $\bar{x}, \bar{y}, \bar{z}, \bar{z}'$ be in $\bar{X}(q)$ such that $\bar{z}$, $\bar{z}'$ have period q. Then $\bar{x}\bar{z} = \bar{y}\bar{z}'$ implies that $\bar{x} = \bar{y}$ and $\bar{z} = \bar{z}'$.

If there exists no split factor $\neq \bar{0}$ and of period r in a sequence $\bar{x}$, $\bar{x}$ is said to be r-irreducible. Then:

(11) If $\bar{x}$ is a sequence of period q^K, there exists a unique splitting $\{\bar{x}_0, \ldots, \bar{x}_K\}$ of $\bar{x}$ such that $\bar{x}_j$ has period q^j for $j = 0, \ldots, K$, and such that $\bar{x}_j$ is q^{j-1}-irreducible for $j = 1, \ldots, K$, $K > 0$. This splitting is called the canonical splitting of $\bar{x}$.

If $\bar{x}$ is a periodic sequence and $(\bar{x})_{n+sp} = 1$ for all s, then p is called a period for the 1 in position n of $\bar{x}$. If in addition for all p' dividing p, $p' \neq p$, $(\bar{x})_{n+sp'} = 0$ for at least one s, then p is called a minimal period of the 1 in position n. The 1 in position n may have more than one minimal period unless the minimal period of the sequence $\bar{x}$ is a power of a prime. For instance, the 1 in position 0 of the sequence $(1, 0, 1, 1, 1, 0, \ldots)$ of period 6 has minimal periods 2 and 3 because the sequences $(1, 0, \ldots)$ of period 2 and $(1, 0, 0, \ldots)$ of period 3 are split factors of $(1, 0, 1, 1, 1, 0, \ldots)$.

(12) If $\bar{x}, \bar{y}$ is in $\bar{X}(q)$ with minimal period q^K, q, respectively, then $\bar{x}\bar{y}$ is q^K-irreducible. If q' is a divisor of q, then $\bar{x}\bar{y}$ is $q^K q'$-irreducible if and only if no 1 in $\bar{y}$ has period q'.

Example. The canonical splitting of the sequence $(0, 1, 1, 1, 1, 1, 1, 1, \ldots)$ of period 8 consists of the sequences $(0, 1, \ldots)$ of period 2, $(0, 0, 1, 0, \ldots)$ of period 4, and $(0, 0, 0, 0, 1, 0, 0, 0, \ldots)$ of period 8. According to (12) the composition

$$(0, 1, 1, 1, 1, 1, 1, 1, \ldots)(0, 1, \ldots) = (0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, \ldots)$$

is 2^k-irreducible for $k = 0, 1, 2, 3$. This is easily verified since no 1 in the sequence has minimal period different from 16.

We shall give short proofs of the previous statements. First, we prove that the composition of sequences defined in Section 4 is associative. If $\bar{x} = (x_0, \ldots, x_n, \ldots)$ is any sequence write $(\bar{x})_n = x_n$. Then if $\bar{x}, \bar{y}, \bar{z}$ are binary sequences $((\bar{x}\bar{y})\bar{z})_n = (\bar{x}\bar{y})_n(z_p)$ where $p = \sum_{i<n} (\bar{x}\bar{y})_i = \sum_{j<q} y_j$ and $q = \sum_{k<n} x_k$. This gives

$$((\bar{x}\bar{y})\bar{z})_n = (x_n)(y_q)(z_p).$$

If any summation is over an empty set of indices, the value is to be interpreted as zero according to the definition of $G(0)$. On the other hand:

$$(\bar{x}(\bar{y}\bar{z}))_n = (x_n)(\bar{y}\bar{z})_q = (x_n)(y_q)(z_p).$$

Since this is valid for all n, $(\bar{x}\bar{y})\bar{z} = \bar{x}(\bar{y}\bar{z})$ for all $\bar{x}, \bar{y}, \bar{z}$. Thus, the composition is associative.

Proof of (1). $(\bar{1}\bar{x})_n = 1 \cdot x_{G(n)} = x_n$ since $G(n) = n$ in this case.

$$(\bar{x}\bar{1})_n = (x_n)(\bar{1})_{G(n)} = x_n \qquad \text{since} \quad (\bar{1})_{G(n)} = 1 \qquad \text{for all } n.$$

Thus, $\bar{1}\bar{x} = \bar{x}\bar{1} = \bar{x}$ for all sequences $\bar{x}$.

Proof of (2). That $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$ implies $\bar{x}\bar{y} = \bar{0}$ is obvious from the definitions. Suppose $\bar{x} \neq \bar{0}$ and $y_j \neq 0$. Then there exists an index k such that $x_k \neq 0$ and $\sum_{i<k} x_i = j$. Then, $(\bar{x}\bar{y})_k \neq 0$.

Proof of (3).

$$\begin{aligned}(\bar{x}(\bar{y} + \bar{z}))_j &= (x_j)(\bar{y} + \bar{z})_{G(j)} \\ &= x_j(y_{G(j)} + z_{G(j)}) = (x_j y_{G(j)}) + (x_j y_{G(j)}).\end{aligned}$$

But this is equal to $((\bar{x}\bar{y}) + (\bar{x}\bar{z}))_j$.

Proof of (4). Let $\bar{x}$ and $\bar{y}$ have period p, q, respectively. Then $(\bar{x}\bar{y})_j = 1$ if and only if $x_j = y_{G(j)} = 1$. In the common period pq this is seen to occur $(w(\bar{x}, pq)/q)\, w(\bar{y}, q)$ times. Thus,

$$w(\bar{x}\bar{y}, pq) = w(\bar{x}, p)\, w(\bar{y}, q) \qquad \text{or} \qquad \bar{w}(\bar{x}\bar{y}) = \bar{w}(\bar{x})\, \bar{w}(\bar{y}).$$

Proof of (5). Let $\bar{x}, \bar{y}$ be any periodic sequences. Then $(\bar{x}\bar{y})_j = x_j y_{G(j)} = 1$ implies $x_j = 1$. Put $\bar{z} = \bar{x} + \bar{x}\bar{y}$, then $z_j = 1$ implies $x_j = 1$ and $(\bar{x}\bar{y})_j = 0$. Then $\{\bar{x}\bar{y}, \bar{z}\}$ is a splitting of $\bar{x}$.

Proof of (6). Suppose $\{\bar{x}_0, ..., \bar{x}_n\}$ is a splitting. Then by (5) $\bar{x}_i \bar{y}_i$ is a split factor in $\bar{x}_i$ for any sequence $\bar{y}_i$. For each index i, let $\{\bar{x}_i \bar{y}_i, \bar{z}_i\}$ be a splitting of $\bar{x}_i$. Then

$$\bar{w}\left(\sum_i \bar{x}_i\right) = \bar{w}\left(\sum_i (\bar{x}_i\bar{y}_i + \bar{z}_i)\right) = \bar{w}\left(\sum_i \bar{x}_i\bar{y}_i\right) + \bar{w}\left(\sum_i \bar{z}_i\right).$$

But since $\{\bar{x}_0, ..., \bar{x}_n\}$ is a splitting

$$\bar{w}\left(\sum \bar{x}_i\right) = \sum \bar{w}(\bar{x}_i) = \sum \bar{w}(\bar{x}_i\bar{y}_i) + \sum \bar{w}(\bar{z}_i) = \sum \bar{w}(\bar{x}_i\bar{y}_i) + \bar{w}\left(\sum \bar{z}_i\right),$$

since obviously $\{\bar{z}_i\}_i$ is a splitting. Thus,

$$\bar{w}\left(\sum \bar{x}_i\bar{y}_i\right) = \sum \bar{w}(\bar{x}_i\bar{y}_i) \qquad \text{and} \qquad \{\bar{x}_i\bar{y}_i\}_i \text{ is a splitting.}$$

Proof of (7). Follows easily by (3) and (4).

Proof of (8). Suppose $\bar{x}_i$ has a period q^{K_i} such that

$$w(\bar{x}_i, q^{K_i}) = p_i + qL_i, (p_i, q) = 1 \qquad \text{for} \quad i = 0,..., n.$$

Then

$$\begin{aligned} w(\bar{x}_0 \cdots \bar{x}_n, q^{K_0+\cdots+K_n}) &= w(\bar{x}_0, q^{K_0}) \cdots w(\bar{x}_n, q^{K_n}) \\ &= (p_0 + qL_0) \cdots (p_n + qL_n) = p_0 \cdots p_n + q(\text{something}), \end{aligned}$$

where $(p_0 \cdots p_n, q) = 1$. Thus, $\bar{x}_0 \cdots \bar{x}_n$ is in $\bar{X}(q)$.

Proof of (9). Let $\bar{x}_i$ be as in the proof of (8) and let I be the only index such that $K_I = \max_i\{K_i\}$. Then

$$\begin{aligned} q^{K_I}\bar{w}(\bar{x}_0 + \cdots + \bar{x}_n) &= q^{K_I}(\bar{w}(\bar{x}_0) + \cdots + \bar{w}(\bar{x}_n)) \\ &= \sum q^{K_I}\bar{w}(\bar{x}_i) = \sum q^{K_I}w(\bar{x}_i, q^{K_i})/q^{K_i} \\ &= w(\bar{x}_I, q^{K_I}) + \sum_{i\neq I} q^{K_I-K_i}w(\bar{x}_i, q^{K_i}), \end{aligned}$$

where $(w(\bar{x}_I, q^{K_I}), q) = 1$. Thus, $\bar{x}_0 + \cdots + \bar{x}_n$ is in $\bar{X}(q)$.

Let $\bar{x}$ be a sequence in $\bar{X}(q)$ of minimal period q^K. Then $w(\bar{x}, q^K)$ is prime to q. Let $\bar{y}_k$ be the sequence of minimal period q defined by: $(\bar{y}_k)_j = 1$ if and only if j is congruent to k mod q. The minimal period of $\bar{z} = \bar{x}\bar{y}_k$ is q^{K+1}, and we shall write

$$\bar{z} = (z_0^{1(k)},..., z_{q^K-1}^{1(k)}, z_0^{2(k)},..., z_{q^K-1}^{2(k)},..., z_0^{q(k)},..., z_{q^K-1}^{q(k)},...).$$

LEMMA 1. *Let $\bar{x}$ be a sequence in $\bar{X}(q)$ of minimal period q^K, and let $\bar{y}_k$ be defined as above. If $(\bar{x})_j = 1$ for some j, $0 \leqslant j \leqslant q^K - 1$, then $z_j^{i(k)}$ (defined previously) is 1 for exactly one $i(k)$, $1 \leqslant i \leqslant q$. Further, if $z_j^{i(k)} = z_j^{p(k')} = 1$, then i is congruent to p* mod q', *a divisor of q, if and only if k is congruent to k'* mod q'.

Proof. By definition $(\bar{z})_j = (\bar{x})_j \cdot (\bar{y}_k)_{G(j)}$, where $G(j) = \sum_{i<j} (\bar{x})_i$. Thus, $(\bar{z})_j = 1$ if and only if $(\bar{x})_j = 1$ and $\sum_{i<j} (\bar{x})_i$ is congruent to k mod q. The numbers $\sum_{i<j+q^K s} (\bar{x})_i$, $0 \leqslant s \leqslant q - 1$, take all values mod q exactly once because $w(\bar{x}, q^K)$ is prime to q. Hence, if $(\bar{x})_j = 1$ there is exactly one i, $1 \leqslant i \leqslant q$, such that $z_j^{i(k)} = 1$. If $z_j^{i(k)} = z_j^{p(k')} = 1$, then $\sum_{i<j+q^K i(k)} (\bar{x})_i$ is congruent to k mod q and $\sum_{i<j+q^K p(k')} (\bar{x})_i$ is congruent to k' mod q. Hence, we have that $\{i(k) - p(k')\}\, w(\bar{x}, q^K)$ is congruent to $k - k'$ mod q. Since

$w(\bar{x}, q^K)$ is prime to q, it follows that i is congruent to p mod q' if and only if k is congruent to k' mod q' where q' is a divisor of q.

Proof of (10). Let $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{z}'$ be in $\bar{X}(q)$ such that $\bar{z}$, $\bar{z}'$ have period q. If $\bar{x}$ has minimal period q^K, then q^{K+1} is the minimal period of $\bar{x}\bar{z}$. Thus, if $\bar{x}\bar{z} = \bar{y}\bar{z}'$, $\bar{x}$ and $\bar{y}$ have the same minimal period. Suppose that $x_j = 1$, while $y_j = 0$ for some index $0 \leqslant j < q^K$. Since the sequence $\bar{z}$ in an obvious way has a splitting in sequences with only one nonzero element in the period q, it follows from (7) and the previous lemma that $(\bar{x}\bar{z})_{j+tq^K} = 1$ for some index t while $(\bar{y}\bar{z}')_{j+tq^K} = 0$ for all t since by (5) $\bar{y}\bar{z}'$ is a split factor in $\bar{y}$. If $\bar{x} = \bar{y}$ and $\bar{x}\bar{z} = \bar{y}\bar{z}'$, then $\bar{x}(\bar{z} - \bar{z}') = \bar{0}$, giving $\bar{z} = \bar{z}'$ since $\bar{x} \neq \bar{0}$ when $\bar{x}$ is in $\bar{X}(q)$. The proof is then complete.

Proof of (11). Let $\bar{x}_0$ be the split factor in $\bar{x}$ of period 1 and maximal frequency $\bar{w}(\bar{x}_0)$. Put $\bar{x}' = \bar{x} + \bar{x}_0$. Then let $\bar{x}_1$ be the split factor in $\bar{x}'$ of period q and maximal frequency etc. It is easy to see that $\{\bar{x}_0 ,..., \bar{x}_K\}$ fulfils the requirements of (11). Suppose $\{\bar{x}_0 ,..., \bar{x}_K\}$, $\{\bar{y}_0 ,..., \bar{y}_K\}$ are two splittings of the type required. Let i be the smallest index such that $\bar{x}_i \neq \bar{y}_i$, and suppose $(\bar{x}_i)_j = 1$ while $(\bar{y}_i)_j = 0$. In the period q^K, $\bar{x}_i$ and $\bar{z} = \sum \bar{x}_j = \sum \bar{y}_j$ then has a 1 in all positions $j + nq^i$, $n = 0,..., q^{K-i} - 1$, altogether q^{K-i} 1's. Thus, $\bar{y}_{i+1} + \cdots + \bar{y}_K$ has 1's in these q^{K-i} positions. Suppose that $\bar{y}_{k+1}$ has $p_{k+1}q^{K-k-1}$ 1's in the positions mentioned, in the period q^K, $k = i,..., K - 1$. Since $\{\bar{y}_{i+1} ,..., \bar{y}_K\}$ is a splitting, the total number of 1's in the period q^K is $p_{i+1}q^{K-i-1} + \cdots + p_K = P$. Since $\bar{y}_i$ is supposed to be q^{i-1}-irreducible for $i = 1,..., K$, we must have

$$\begin{aligned} p_{i+1} &< q, \\ p_{i+2} &< q^2 - p_{i+1}q, \qquad \text{etc.}, \\ p_K &< q^{K-i} - p_{i+1}q^{K-i-1} - \cdots - p_{K-1}q. \end{aligned}$$

By proving that the maximum of P is obtained by choosing $p_k = q - 1$ for $k = i + 1,..., K$, one obtains that P is less than or equal to

$$(q - 1)(q^{K-i-1} + \cdots + q + 1) = q^{K-i} - 1,$$

contradicting that $\bar{z}$ has q^{K-i} 1's in the required positions in the period q^K. Thus, the splittings are necessarily equal.

Proof of (12). Suppose $\bar{x}$, $\bar{y}$ is in $\bar{X}(q)$ with minimal period q^K, q respectively. By Lemma 1 no 1 in $\bar{x}\bar{y}$ has period q^K since $\bar{y} \neq \bar{1}$. Further q^Kq', when q' is a divisor of q, is a period of a 1 in $\bar{x}\bar{y}$ (Lemma 1) if and only if q' is a period of a 1 in $\bar{y}$.

5. Applications to Composite Automata

In this section we investigate the output structure of the type of composite automata described in Section 3. All the component automata are of the type introduced in Section 2. An example is worked out in Section 6. We are aiming at a complete description of the output sequence of the composite automaton with regard to the frequency of 1's and 0's in all periodic subsequences. In order to obtain this, we introduce sufficient restrictions on the connections and component automata to make the algebraic analysis of Section 4 applicable. The set of sequences associated with the edges going out from any vertex in the connection graph is supposed to be a splitting. Then we can calculate the frequencies of the sequences involved by the definition of splitting and property 4 of Section 4. All component automata except $\{\bar{x}_{it}\}_i$ are supposed to be represented by sequences in $\bar{X}(q)$ of minimal period q. Then, if the connection graph is suitably designed, the properties (8) and (9) of Section 4 apply. It then follows that all sequences involved are contained in the class $\bar{X}(q)$. Properties (11) and (12) of Section 4 make it possible to find all periodic subsequences of the output sequence and calculate the frequencies of 1's and 0's (Theorems 1 and 2). Finally, (10) of Section 4 indicates the fact proved in Theorem 3, that there is a one-to-one correspondence between composite automata (with distinguished initial state) of the type considered and the generated output sequences. We proceed with the details.

Let G be a connection graph of order K with connection matrix $\bar{G} = ((\bar{x}_{ij}))$. From now on, suppose that for all indices i, j, $\bar{x}_{ij} \neq \bar{1}$ and has q as period, q being a fixed integer. If (v_i, v_j) is an edge in G with $v_j \neq v_t$, we suppose that $\bar{x}_{ij} \in \bar{X}(q)$. For all indices $i \neq t$, $\{\bar{x}_{ij}\}_j$ is supposed to be a splitting. If $g = ((v_{i_1}, v_{i_2}), \ldots, (v_{i_n}, v_{i_{n+1}}))$ is any path in G, let

$$w(g) = w(\bar{x}_{i_1 i_2}, q) \cdots w(\bar{x}_{i_n i_{n+1}}, q)$$

and

$$\bar{w}(g) = \bar{w}(\bar{x}_{i_1 i_2}) \cdots \bar{w}(\bar{x}_{i_n i_{n+1}}).$$

With $\bar{w}_{ij} = \bar{w}(\bar{x}_{ij})$, the matrix $((\bar{w}_{ij}))$ is denoted by $\bar{w}(\bar{G})$. The length (the number of edges) of the path g is denoted by L_g.

Lemma 2. *Let G be a connection graph with associated matrix $((\bar{x}_{ij}))$. If v is a vertex on level L in G, then q^L is a period of $\bar{x}(v)$ and*

$$w(\bar{x}(v), q^L) = \sum_g q^{L-L_g} w(g),$$

summation over all paths from v_0 to v in G.

Proof. By induction on the level it can be proved that if (v_i, v_j) and (v_m, v_n) are any pair of edges not on the same path, then $\{\bar{x}(v_i, v_j), \bar{x}(v_m, v_n)\}$ is a splitting. In particular, if v is any vertex in G, then $\{\bar{x}(v_i, v)\}_i$ is a splitting. Now, suppose v_i is a vertex on level L_i and (v_i, v) is an edge in G. Suppose Lemma 2 is true for the vertex v_i, then $w(\bar{x}(v_i), q^{L_i}) = \sum_g q^{L_i - L_g} w(g)$, summation over all paths from v_0 to v_i. Then

$$w(\bar{x}(v_i, v), q^{L_i+1}) = w(\bar{x}(v_i), q^{L_i})\, w(\bar{x}_{ij}, q) \qquad \text{if} \quad v = v_j .$$

Thus,

$$\begin{aligned} w(\bar{x}(v_i, v), q^L) &= q^{L-L_i-1} \sum_g q^{L_i - L_g} w(g)\, w(\bar{x}_{ij}, q) \\ &= \sum_g q^{L-(L_g+1)}\, w(g)\, w(\bar{x}_{ij}, q) \\ &= \sum_{g'} q^{L-L_{g'}} w(g') \end{aligned}$$

where the last summation is over all paths g' from v_0 to v containing the edge (v_i, v). By considering all vertices v_i such that (v_i, v) is an edge, the result follows by induction on the level since $\{x(v_i, v)\}_i$ is a splitting.

COROLLARY 1. *Let $v \neq v_t$ be a vertex on level L in G, and suppose there is only one path in G from v_0 to v of length L. Then, $w(\bar{x}(v), q^L)$ is prime to q, and, thus, q^L is the minimal period of $\bar{x}(v)$.*

Proof. If $g = ((v_0, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_{L_g}-1}, v))$ is a path from v_0 to v, then $w(g)$ is prime to q, because all edge sequences on the path are in $\bar{X}(q)$. Since $w(\bar{x}(v), q^L) = \sum_g q^{L-L_g} w(g)$, summation over all paths from v_0 to v, and only one path has length L, the number $w(\bar{x}(v), q^L)$ is prime to q. Hence, the minimal period of $\bar{x}(v)$ is q^L.

THEOREM 1. *Let v_i be a vertex on level L in g. Then*

$$\bar{w}(\bar{x}(v_i)) = \Big(\sum_{k=1}^{L} \bar{w}(\bar{G})^k\Big)_{0i} .$$

Proof. $\bar{w}(\bar{G})$ is a matrix with elements in the field of rational numbers, and the matrix operations are defined as usual. By the previous lemma $\bar{w}(\bar{x}(v_i)) = \sum_g \bar{w}(g)$, summation over all paths from v_0 to v_i. Comparison with the properties of the adjacency matrix of a graph (Berge, 1964, Chapter 14) gives the result.

A connection graph is called pseudoproper if for all vertices $v \neq v_t$ there is only one path from v_0 to v of length equal to the level of v. If G is pseudoproper, so are the subgraphs G_i (defined in Section 3) for all vertices v_i.

LEMMA 3. *Let G be a pseudoproper connection graph with connection matrix $((\bar{x}_{ij}))$ and let v be a vertex in G different from v_0 and v_t. Then all sequences $\bar{x}(v)$ and $\bar{x}(v_i, v)$ are in the set $\bar{X}(q)$.*

Proof. Lemma 3 is trivially true for all graphs of order 0 and 1. Suppose that Lemma 3 is true for all pseudoproper connection graphs of order less than $K > 1$ and let G be a pseudoproper graph of order K. If $v_i \neq v_0$ is a vertex on level $K - 2$ or less, it follows from the induction hypothesis that $\bar{x}(v_i)$ is in $\bar{X}(q)$, and $\bar{x}(v_i, v_j) = \bar{x}(v_i)\,\bar{x}_{ij}$ is in $\bar{X}(q)$ by property 8 of Section 4 if $j \neq t$. $\bar{x}(v_0, v_j)$ is also in $\bar{X}(q)$ if $j \neq t$. In particular, if v_k is a vertex on level $K - 1$, then all sequences $\{\bar{x}(v_p, v_k)\}_p$ are in $\bar{X}(q)$ and the set $\{\bar{x}(v_p, v_k)\}_p$ of sequences is a splitting of $\bar{x}(v_k)$. Since the graph is pseudoproper, property 9 of Section 4 applies such that $\bar{x}(v_k) = \sum_p \bar{x}(v_p, v_k)$ is in $\bar{X}(q)$. This completes the proof.

THEOREM 2. *Let G be a pseudoproper connection graph with matrix $((\bar{x}_{ij}))$, and let $\{v_i\}_i$ be the vertices on colevel 1 and level L in G. For each i, let $\bar{x}_{it}^{q'}$ be the split factor of $\bar{x}_{it}$ containing all 1's of minimal period q'. Then $\{x(v_i)\,\bar{x}_{it}^{q'}\}_i$ is a splitting of the split factor of $\bar{x}(v_t)$ consisting of all 1's of minimal period $q^L q'$, $1 < q' \leqslant q$.*

Theorem 2 characterizes all periods of the 1's in the response sequence $\bar{x}(v_t)$ of $\bar{G}$. By applying Theorem 2 to the graph G' with connection matrix $((\bar{x}'_{ij}))$, (defined below), generating the sequence $\bar{1} + \bar{x}(v_t)$, the periods of the 0's in $\bar{x}(v_t)$ are determined. Theorem 2 is most easily proved by using the graph G', so we proceed to define this graph before proving Theorem 2.

Let G be a connection graph with connection matrix $((\bar{x}_{ij}))$. The graph G' has the same vertices as G, and if $v_j \neq v_t$ and (v_i, v_j) is an edge in G, then (v_i, v_j) is also an edge in G'. Further, (v_i, v_t) is an edge in G' if and only if $\sum_k \bar{x}_{ik} \neq \bar{1}$, summation over all outgoing edges from v_i in G. Clearly G' is a connection graph of the same order as G, and G' is pseudoproper if G is pseudoproper.

A connection matrix $((\bar{x}'_{ij}))$ for G' is defined as follows: If $j \neq t$ and (v_i, v_j) is an edge in G', then $\bar{x}'_{ij} = \bar{x}_{ij}$. If (v_i, v_t) is an edge in G', then $\bar{x}'_{it} = \bar{1} + \sum_k \bar{x}_{ik}$, summation over all outgoing edges from v_i in G.

LEMMA 4. *If $\bar{x}'(v_t)$ is the response sequence of the composite automaton $((\bar{x}'_{ij}))$, then $\{\bar{x}'(v_t), \bar{x}(v_t)\}$ is a splitting of $\bar{1}$.*

Proof. Suppose $(\bar{x}(v_t))_n = 0$ and let v_i be the unique vertex on highest level in G with $(\bar{x}(v_i))_n = 1$. If $\bar{y}$ is the sequence of period q with $w(\bar{y}, q) = 1$ and $(\bar{x}(v_i)\,\bar{y})_n = 1$, then $\bar{y}$ is not a split factor in $\bar{x}_{ij}$ for any edge (v_i, v_j) in G. $\bar{y}$ is, therefore, a split factor in $\bar{x}'_{it}$. Hence, $(\bar{x}'(v_t))_n = 1$. This completes the proof since $G = G''$.

Proof of Theorem 2. Let $q' \neq 1$ be a divisor of q. Suppose $(\bar{x}(v_t))_n = 1$, and let L be the highest level such that for some vertex v_i on level L $(\bar{x}(v_i))_{n+sq^L} = 1$ for all s. Let $\bar{y}_k$ be the sequence of period q with $(\bar{y}_k)_k = 1$ and $w(\bar{y}_k, q) = 1$. If $(\bar{x}(v_i)\,\bar{y}_{k_0})_n = 1$, then $(\bar{x}(v_i)\,\bar{x}_{it})_{n+q'q^L s} = 1$ for all s if and only if the sequences $\bar{y}_{k_0+rq'}$, $0 \leqslant r < q/q'$ are split factors of $\bar{x}_{it}$ by Lemma 1. If for some r $\bar{y}_{k_0+rq'}$ is not a split factor in $\bar{x}_{it}$, then let v_j be the vertex in G' with split factor $\bar{y}_{k_0+rq'}$ of $\bar{x}'_{ij}$. Let $((v_j, v_{j_1}), (v_{j_1}, v_{j_2}),\ldots, (v_{j_n}, v_t))$ be a path in G' from v_j to v_t. By Lemma 1 we have that $(\bar{x}'(v_j, v_{j_1}))_{n+q'q^L s'} = 1$ for at least one s', $(\bar{x}'(v_{j_1}, v_{j_2}))_{n+q'q^L s''} = 1$ for at least one s'' etc. Hence, $(\bar{x}'(v_t))_{n+q'q^L p} = 1$ for at least one p, and $q'q^L$ is not a period of the 1 in position n of $\bar{x}(v_t)$. Hence, it is proved that $\{\bar{x}(v_i)\,\bar{x}_{it}^{q'}\}_i$, v_i on level L and colevel 1 is a splitting of the split factor of $\bar{x}(v_t)$ consisting of all 1's of minimal period $q'q^L$.

COROLLARY 2. *Let G be a pseudoproper connection graph of order K with connection matrix $((\bar{x}_{ij}))$. Then the response sequence $\bar{x}(v_t)$ has minimal period $q^{K-1}p$ when p is the least common multiple of the minimal periods of the sequences $\bar{x}_{it}$.*

A connection graph G is called proper if for every triple of different vertices (v_i, v_j, v_k) such that (v_i, v_k) and (v_j, v_k) are edges in G, the vertices v_i and v_j are on different levels. In particular a proper graph is pseudoproper.

THEOREM 3. *If $\bar{x}$ is the response sequence of a composite automaton with proper connection graph, then $\bar{x}$ uniquely determines the proper connection graph and the connection matrix.*

Proof. Introduce a vertex v_t, and let the response sequence be $\bar{x}(v_t) = \bar{x}$. Denote the canonical splitting of $\bar{x}(v_t)$ by $\{\bar{x}(v_i, v_t)\}_i$ such that $\bar{x}(v_i, v_t)$ has period q^{L_i+1} and is q^{L_i}-irreducible. For each index i, introduce a vertex v_i. Then the vertices on colevel 1 are given since the vertices on colevel 1 are on different levels when G is proper. By (10) of Section 4, there exist uniquely determined sequences $\bar{x}(v_i)$, $\bar{x}_{it}$, for each index i, such that $\bar{x}(v_i)$ is in $\bar{X}(q)$, $\bar{x}_{it}$ has period q and $\bar{x}(v_i)\,\bar{x}_{it} = \bar{x}(v_i, v_t)$. Thus, the sequences $\{\bar{x}_{it}\}_i$ are determined. We then continue the reconstruction of the graph G and matrix

$((\bar{x}_{ij}))$ in the following way. Find the canonical splitting of $\bar{x}(v_t)$. Since the graph is proper, a vertex v_j and an edge (v_j, v_i) has to be introduced for each split factor in the canonical splitting of $\bar{x}(v_i)$. We then use (10) of Section 4 to get the sequences $\bar{x}(v_j)$ and $\bar{x}_{ji}$ such that $\bar{x}(v_j)\,\bar{x}_{ji}$ is equal to the split factor considered. Note that if a vertex v_j is introduced such that the sequence $\bar{x}(v_j)$ is equal to a sequence $\bar{x}(v_k)$ where v_k is a vertex already introduced, the vertices v_j and v_k are identified. Then continue to construct the vertices on colevel 3, colevel 4 etc. until the whole graph and the connection matrix is reconstructed.

COROLLARY 3. *Suppose q is a prime number. Then if G or G' is a proper graph, $((\bar{x}_{ij}))$ is uniquely determined by $\bar{x}(v_t)$.*

Proof. If G is not proper, $((\bar{x}'_{ij}))$ is uniquely determined by the sequence $\bar{1} + \bar{x}(v_t) = \bar{x}'(v_t')$, and $((\bar{x}_{ij}^T))$ is determined by $((\bar{x}_{ij}'^T))$ since $G = G''$. Let G be a connection graph with matrix $((\bar{x}_{ij}))$. To every sequence $\bar{x}_{ij}$ corresponds an automaton of the type considered in Section 2 with distinguished initial state. The set of states of the composite automaton is by definition the product of the sets of states of the components. Thus, the matrix $((\bar{x}_{ij}))$ specifies a distinguished initial state for the composite automaton. For any sequence $\bar{x} = (x_0, x_1, ...)$, let $C_k(\bar{x}) = (x_k, x_{k+1}, ...)$ be the sequence obtained by translating $\bar{x}$ k steps, $k = 0, 1, ..., q - 1$.

Then, for each vertex $v_i \neq v_t$, $\{C_{k(i)}(\bar{x}_{ij})\}_j$ is a splitting. Here, $k(i)$ is an integer dependent on i. If $\bar{x}_{ij}$ is the response sequence of the automaton a_{ij} in initial state s_0 (input sequence $\bar{1}$), then $C_k(\bar{x}_{ij})$ is the response sequence of the same automaton in initial state s_k. It follows that the matrix $((C_{k(i)}(\bar{x}_{ij})))$ is a connection matrix for the same graph G specifying the same composite automaton with another distinguished initial state s'. The set of states s' obtained by different choice of values for $k(i)$, specifies a subautomaton A' of A. A' is called the reduced composite automaton. Two states are by definition in the same state component, if there is a finite input sequence which takes the automaton from one to the other state.

THEOREM 4. *Let G be a pseudoproper connection graph of order K and with L vertices. Suppose $((\bar{x}_{ij}))$ is an associated connection matrix such that all $\bar{x}_{ij}$ are in $\bar{X}(q) \cap \{\bar{0}\}$. Then the reduced composite automaton A' has q^{L-K-1} state components.*

Proof. By Corollary 2, $\bar{x}(v_t)$ has minimal period q^K. This means that with A' in any initial state, the response sequence to the input sequence $\bar{1}$ has minimal period q^K, such that each state component contains q^K elements.

Since the state set of A' has q^{L-1} elements, there are q^{L-K-1} state components of A'.

COROLLARY 4. *Let G be a proper connection graph such that all sequences $\bar{x}_{ij}$ in the connection matrix $((\bar{x}_{ij}))$ are in $\bar{X}(q) \cup \{\bar{0}\}$. Then the reduced composite automaton A' is minimal.*

Proof. To prove that A' is minimal is equivalent to prove that there exists an input sequence giving different output sequences to different initial states [Both, 1968, p. 82]. By Theorem 3, different initial states of A' generate different output sequences to the input sequence $\bar{1}$, because different initial states correspond to different connection matrices.

6. AN EXAMPLE

Let $A_1, A_2, A_3, \ldots, A_9$ be component automata of the type specified in Section 2 and A the composite automaton given in Fig. 1. In our theory, the

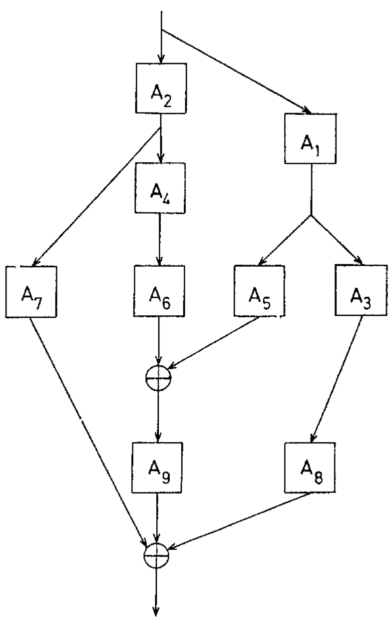

FIG. 1. The composite automaton A.

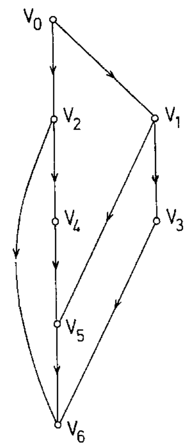

FIG. 2. The graph G.

interconnections are represented by the graph G in Fig. 2. Notice that G is a proper connection graph of order 4. Let the automata A_1 ,..., A_9 have response sequences all of period 3:

$$\begin{aligned}
A_1 &= \bar{x}_{01} = (1, 0, 1,...),\\
A_2 &= \bar{x}_{02} = (0, 1, 0,...),\\
A_3 &= \bar{x}_{13} = (1, 0, 0,...),\\
A_4 &= \bar{x}_{24} = (1, 1, 0,...),\\
A_5 &= \bar{x}_{15} = (0, 0, 1,...),\\
A_6 &= \bar{x}_{45} = (0, 1, 1,...),\\
A_7 &= \bar{x}_{26} = (0, 0, 1,...),\\
A_8 &= \bar{x}_{36} = (1, 0, 1,...),\\
A_9 &= \bar{x}_{56} = (0, 0, 1,...).
\end{aligned}$$

This gives the frequency matrix

$$\bar{w}(\bar{G}) = 1/3 \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The frequency of the sequences generated at the vertices is then given by $\bar{w}(\bar{x}(v_i)) = (\sum_{k=1}^{4} \bar{w}(\bar{G})^k)_{0i}$:

i	0	1	2	3	4	5	6
$\bar{w}$	1	2/3	1/3	$2/3^2$	$2/3^2$	$10/3^3$	$31/3^4$

The vertices on colevel 1 in G are v_2, v_3, and v_5. These are on levels 1, 2, and 3, respectively.

The canonical splitting of $\bar{x}(v_t)$ is then given by the sequences $\bar{x}(v_2)\,\bar{x}_{26}$, $\bar{x}(v_3)\,\bar{x}_{36}$, and $\bar{x}(v_5)\,\bar{x}_{56}$ of frequency $1/3^2$, $4/3^3$ and $10/3^4$, respectively. This means that if the automaton A is clocked regularly, the response sequence $\bar{x}(v_6)$ has minimal period 3^4. This sequence has subperiods 3^2, 3^3, and 3^4 of 1's and no others. If $\bar{x}(v_6)$ is split as a sum of sequences of these shorter periods:

$$\begin{aligned}\bar{x}(v_6) &= \bar{x}_2 + \bar{x}_3 + \bar{x}_5 \\ &= \bar{x}(v_2)\,\bar{x}_{26} + \bar{x}(v_3)\,\bar{x}_{36} + \bar{x}(v_5)\,\bar{x}_{56}\,.\end{aligned}$$

Then $\bar{x}_2$ has one 1 in its minimal period 3^2, $\bar{x}_3$ has 4 1's in its minimal period 3^3 and $\bar{x}_5$ has 10 1's in its minimal period 3^4.

Subperiods of 0's in $\bar{x}(v_6)$ are determined by the graph G' in Fig. 3. The

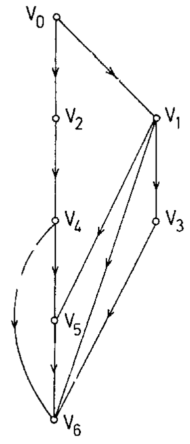

FIG. 3. The graph G'.

vertices on colevel 1 in G' are v_1, v_3, v_4 and v_5 on levels 1, 2, 2, and 3 respectively. Thus, the sequence $\bar{x}(v_6)$ has 0's of periods 3^2, 3^3 and 3^4, and no 0 of minimal period different from these. From Theorem 4, the composite automaton has at least $3^{7-4-1} = 3^2$ state components generating different cycles since G is proper. The cycles generated by these states, have the same minimal period 3^4.

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